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Research on a Fractional Integral of Fractional Rational Function

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Abstract: **In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, a fractional integral of fractional rational is studied. In addition, our result is a generalization of classical calculus result.**

Keywords: **Jumarie's modified R-L fractional calculus, new multiplication, fractional analytic functions, fractional integral, fractional rational function.**

I. INTRODUCTION

Fractional calculus belongs to the field of mathematical analysis, involving the research and applications of arbitrary order integrals and derivatives. Fractional calculus originated from a problem put forward by L'Hospital and Leibniz in 1695. Therefore, the history of fractional calculus was formed more than 300 years ago, and fractional calculus and classical calculus have almost the same long history. Since then, fractional calculus has attracted the attention of many contemporary great mathematicians, such as N. H. Abel, M. Caputo, L. Euler, J. Fourier, A. K. Grunwald, J. Hadamard, G. H. Hardy, O. Heaviside, H. J. Holmgren, P. S. Laplace, G. W. Leibniz, A. V. Letnikov, J. Liouville, B. Riemann, M. Riesz, and H. Weyl. With the efforts of researchers, the theory of fractional calculus and its applications have developed rapidly. On the other hand, fractional calculus has wide applications in physics, electrical engineering, , viscoelasticity, control theory, economics, and other fields [1-10].

However, different from the traditional calculus, the rule of fractional derivative is not unique, many scholars have given the definitions of fractional derivatives. The common definition is Riemann-Liouville (R-L) fractional derivatives. Other useful definitions include Caputo fractional derivatives, Grunwald-Letnikov (G-L) fractional derivatives, and Jumarie type of R-L fractional derivatives to avoid non-zero fractional derivative of constant function [11-15].

In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions, we obtain the exact solution of a fractional integral of fractional rational function. In fact, our result is a generalization of ordinary calculus result.

II. PRELIMINARIES

Firstly, we introduce the fractional calculus used in this paper.

Definition 2.1 ([16])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt \,, \tag{1}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

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$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\,, \tag{2}$$

where $\Gamma(\ )$ is the gamma function.

Proposition 2.2 ([17])**:** *If* α, β, x_0, C *are real numbers and* $\beta \ge \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)[(x-x_0)^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{4}$$

Definition 2.3 ([18])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a,b)$, and $0 < \alpha \le 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([19])**:** If $0 < \alpha \le 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}. \tag{6}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{8}$$

III. MAIN RESULT

In this section, we study a fractional integral of fractional rational function.

Theorem 3.1: *Suppose that* $0 < \alpha \le 1$, *then the* α*-fractional integral*

$$\left({}_0I_x^{\alpha}\right)\left[\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-2)}\right] = \frac{1}{2}\cdot arctan_\alpha(x^\alpha) + \frac{1}{2}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes_\alpha\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}\right)^{\otimes_\alpha(-1)}. \tag{9}$$

Proof Let $\frac{1}{\Gamma(\alpha+1)}x^\alpha = tan_\alpha(t^\alpha)$, then $\frac{1}{\Gamma(\alpha+1)}t^\alpha = arctan_\alpha(x^\alpha)$. Therefore,

$$\left({}_0I_x^{\alpha}\right)\left[\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-2)}\right]$$

$$= \left({}_0I_x^{\alpha}\right)\left[\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-2)}\otimes_\alpha\left({}_0D_x^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]\right]$$

$$= \left({}_0I_t^{\alpha}\right)\left[\left[1+\left(tan_\alpha(t^\alpha)\right)^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-2)}\otimes_\alpha\left({}_0D_t^{\alpha}\right)[tan_\alpha(t^\alpha)]\right]$$

$$= \left({}_0I_t^{\alpha}\right)\left[\left[\left(sec_\alpha(t^\alpha)\right)^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-2)}\otimes_\alpha\left(sec_\alpha(t^\alpha)\right)^{\otimes_\alpha 2}\right]$$

$$= \left({}_0I_t^{\alpha}\right)\left[\left(sec_\alpha(t^\alpha)\right)^{\otimes_\alpha(-2)}\right]$$

$$= \left({}_0I_t^{\alpha}\right)\left[\left(cos_\alpha(t^\alpha)\right)^{\otimes_\alpha 2}\right]$$

$$= \left({}_0I_t^{\alpha}\right)\left[\frac{1}{2}[1+cos_\alpha(2t^\alpha)]\right]$$

$$= \frac{1}{2}\left({}_0I_t^{\alpha}\right)[1+cos_\alpha(2t^\alpha)]$$

$$= \frac{1}{2}\cdot\frac{1}{\Gamma(\alpha+1)}t^\alpha+\frac{1}{4}\cdot sin_\alpha(2t^\alpha)$$

$$= \frac{1}{2}\cdot\frac{1}{\Gamma(\alpha+1)}t^\alpha+\frac{1}{2}\cdot sin_\alpha(t^\alpha)\otimes_\alpha cos_\alpha(t^\alpha)$$

$$= \frac{1}{2}\cdot\frac{1}{\Gamma(\alpha+1)}t^\alpha+\frac{1}{2}\cdot tan_\alpha(t^\alpha)\otimes_\alpha\left(cos_\alpha(t^\alpha)\right)^{\otimes_\alpha 2}$$

$$= \frac{1}{2}\cdot\frac{1}{\Gamma(\alpha+1)}t^\alpha+\frac{1}{2}\cdot tan_\alpha(t^\alpha)\otimes_\alpha\left(\left(sec_\alpha(t^\alpha)\right)^{\otimes_\alpha 2}\right)^{\otimes_\alpha(-1)}$$

$$= \frac{1}{2}\cdot\frac{1}{\Gamma(\alpha+1)}t^\alpha+\frac{1}{2}\cdot tan_\alpha(t^\alpha)\otimes_\alpha\left(1+\left(tan_\alpha(t^\alpha)\right)^{\otimes_\alpha 2}\right)^{\otimes_\alpha(-1)}$$

$$= \frac{1}{2}\cdot arctan_\alpha(x^\alpha)+\frac{1}{2}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes_\alpha\left(1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}\right)^{\otimes_\alpha(-1)}.$$

Q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions, we study a fractional integral of fractional rational function. In fact, our result is a generalization of traditional calculus result. In the future, we will continue to use Jumarie's modified R-L fractional calculus and the new multiplication of fractional analytic functions to solve problems in fractional differential equations and applied mathematics.

REFERENCES

[1] V. V. Uchaikin, Fractional Derivatives for Physicists and Engineers, Vol. 1, Background and Theory, Vol. 2, Application. Springer, 2013.

[2] Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, Advanced Engineering Technology and Application, vol. 5, no. 2, pp. 41-45, 2016.

[3] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, Vol. 8, No. 5, 660, 2020.

[4] M. F. Silva, J. A. T. Machado, A. M. Lopes, Fractional order control of a hexapod robot, Nonlinear Dynamics, vol. 38, pp. 417-433, 2004.

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Vol. 10, Issue 1, pp: (42-45), Month: April 2023 – September 2023, Available at: www.paperpublications.org

[5] A. Carpinteri, F. Mainardi, (Eds.), Fractals and fractional calculus in continuum mechanics, Springer, Wien, 1997.

[6] N. Heymans, Dynamic measurements in long-memory materials: fractional calculus evaluation of approach to steady state, Journal of Vibration and Control, vol. 14, no. 9, pp. 1587-1596, 2008.

[7] R. C. Koeller, Applications of fractional calculus to the theory of viscoelasticity, Journal of Applied Mechanics, vol. 51, no. 2, 299, 1984.

[8] T. Sandev, R. Metzler, & Ž. Tomovski, "Fractional diffusion equation with a generalized Riemann–Liouville time fractional derivative," Journal of Physics A: Mathematical and Theoretical, vol. 44, no. 25, 255203, 2011.

[9] J. P. Yan, C. P. Li, On chaos synchronization of fractional differential equations, Chaos, Solitons & Fractals, vol. 32, pp. 725-735, 2007.

[10] C. -H. Yu, A study on fractional RLC circuit, International Research Journal of Engineering and Technology, vol. 7, no. 8, pp. 3422-3425, 2020.

[11] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations; John Willy and Sons, Inc.: New York, NY, USA, 1993.

[12] K. B. Oldham, J. Spanier, The Fractional Calculus; Academic Press: New York, NY, USA, 1974.

[13] Podlubny, Fractional Differential Equations; Academic Press: New York, NY, USA, 1999.

[14] S. Das, Functional Fractional Calculus, 2nd Edition, Springer-Verlag, 2011.

[15] Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, 2010.

[16] C. -H. Yu, Application of differentiation under fractional integral sign, International Journal of Mathematics and Physical Sciences Research, vol. 10, no. 2, pp. 40-46, 2022.

[17] U. Ghosh, S. Sengupta, S. Sarkar and S. Das, Analytic solution of linear fractional differential equation with Jumarie derivative in term of Mittag-Leffler function, American Journal of Mathematical Analysis, vol. 3, no. 2, pp. 32-38, 2015.

[18] C. -H. Yu, Study of fractional analytic functions and local fractional calculus, International Journal of Scientific Research in Science, Engineering and Technology, vol. 8, no. 5, pp. 39-46, 2021.

[19] C. -H. Yu, Exact solutions of some fractional power series, International Journal of Engineering Research and Reviews, vol. 11, no. 1, pp. 36-40, 2023.